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A CHARACTERIZATION OF THE MINIMALBASIS OF THE TORUS

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D. König asks the interesting question in [7] whether there are facts corresponding to the theorem of Kuratowski which apply to closed orientable or non-orientable surfaces of any genus. Since then this problem has been solved only for the projective plane ([2], [3], [8]). In order to demonstrate that König's question can be affirmed we shall first prove, that every minimal graph of the minimal basis of all graphs which cannot be embedded into the orientable surface $\mathfrak{f}$ of genus p has orientable genus $p+1$ and non-orientable genus q with $1 \leqq q \leqq 2p+2$. Then let $\mathfrak{f}$ be the torus. We shall derive a characterization of all minimal graphs of the minimal basis with the non-orientable genus $q=1$ which are not embeddable into the torus. There will be two very important graphs signed with X_8 and X_7 later. Furthermore 19 graphs $G_1, G_2, \ldots, G_{19}$ of the minimal basis $M(\text{torus}, >_4)$ will be specified. We shall prove that five of them have non-orientable genus $q=1$, ten of them have non-orientable genus $q=2$ and four of them non-orientable genus $q=3$. Then we shall point out a method of determining graphs of the minimal basis $M(\text{torus}, >_4)$ which are embeddable into the projective plane. Using the possibilities of embedding into the projective plane the results of [2] and [3] are necessary. This method will be called saturation method. Using the minimal basis $M(\text{projective plane}, >_4)$ of [3] we shall at last develop a method of determining all graphs of $M(\text{torus}, >_4)$ which have non-orientable genus $q \geqq 2$. Applying this method we shall succeed in characterizing all minimal graphs which are not embeddable into the torus. The importance of the saturation method will be shown by determining another graph $G_{20} \neq G_1, G_2, \ldots, G_{19}$ of $M(\text{torus}, >_4)$.

In order to characterize the minimal graphs which are not embeddable into the torus it is necessary to introduce and to explain some concepts and notations. A graph is called subdivision $U(G)$ of a graph[1] G if and only if there are only "new" vertices added on edges of G. It is $H >_1 G$, if and only if there is a subdivision $U(G)$ which is a subgraph of H. As is well known $>_1$ is a partial order relation, it is called subdivision relation. It is remarkable that $>_1$ is embedding-hereditary[2]. That means: If $H >_1 G$ and H is embeddable into the surface $\mathfrak{f}$, then G is embeddable into $\mathfrak{f}$. Let Γ be the set of all graphs which are not embeddable into $\mathfrak{f}$, and $>$ a partial order relation on Γ. Then $M(\Gamma, >)$ or $M(\mathfrak{f}, >)$ denotes the set of all relative $>$ minimal graphs of Γ. It is called the minimal basis of Γ of $\mathfrak{f}$. Using the concept of minimal basis the theorem of Kuratowski tersely says: $M(\text{plane}, >_1) = \{K_5, K_{3,3}\}$. That means: A graph G is not embeddable into the plane if and only

[1] We only consider finite simple graphs without loops. G itself is also called a subdivision of G. By the way we assume the notation of [2], [3] and [4].

[2] X_8 is called "Wagnerscher Graph" in [5]. X_8 is the only non-planar graph of the homomorphic basis of the K_5.

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if G contains a subdivision of K_5 or of $K_{3,3}$. The subdivision relation $>_1$ can be defined in a further way. Therefore the five elementary relations R_0, R_1, R_2, R_3, and R_4 on the set Γ of all finite, undirected simple graphs are introduced. R_0 and R_1 can be thought of as removing an edge or an isolated vertex of a graph $G\in\Gamma$ or contracting an edge $k=(a, b)$ of G where at least one of the vertices a or b has got the degree 2 at most. R_2 means contracting an edge $k'=(c, d)$ where the endpoints c and d are incident to at least three edges of G. R_3 is defined to be a relation which replaces a trihedral $\{a\}*\{e_1, e_2, e_3\}$ of G by the triangle (e_1, e_2, e_3, e_1). Using R_4 we must replace a double-trihedral by a double-triangle. We obtain five partial order relations $>_0, >_1, >_2, >_3, >_4$ with $>_0\subseteqq>_1\subseteqq>_2\subseteqq>_3\subseteqq>_4$ by determining the reflexive, transitive closure

$$>_i = \bigcap_{\mathfrak{R}_i \supseteqq V_i} \mathfrak{R}_i$$

with $i=0, 1, 2, 3, 4$, $V_i=R_0\cup\ldots\cup R_i$, and $\mathfrak{R}_i$ reflexive, transitive relations on Γ. If the set of all relative to $>_i$ minimal graphs is defined to be the minimal basis $M(\Gamma, >_i)$ the following inclusions are true: $M(\Gamma, >_4)\subseteqq M(\Gamma, >_3)\subseteqq M(\Gamma, >_2)\subseteqq$ $\subseteqq M(\Gamma, >_1)\subseteqq M(\Gamma, >_0)$. $>_0$ can be thought of as the well-known subgraph-relation, and $>_1$ and $>_2$ the well—known subdivisionrelation and homomorphic relation. Because of $K_{3,3}>_4K_5$ the theorem of Kuratowski says: $M(\Gamma_0, >_4)=\{K_5\}$, if Γ_0 is the set of all nonplanar graphs of Γ.

First, the following theorem (1.1) is a consequence of the embeddinghereditariness:

(1.1) *A graph G is not embeddable into an orientable or non-orientable surface $\mathfrak{f}$ if and only if G contains a subdivision of at least one graph of the minimal basis $M(\mathfrak{f}, >_1)$.*

Theorem (1.1) distinctly shows the equivalence between König's problem and the determinations of all graphs of the minimal basis $M(\mathfrak{f}, >_1)$. The following theorem will characterize all the graphs of the minimal basis $M(\mathfrak{f}, >_1)$.

(1.2) *A graph H belongs to $M(\mathfrak{f}, >_1)$ if and only if H has the following three properties:*
1. H is not embeddable into $\mathfrak{f}$.
2. For each edge k of H the graph $H-k$ is embeddable into $\mathfrak{f}$.
3. Each vertex e of H is incident with at least three edges of H.

Proof. Let Γ be the set of all graphs which are not-embeddable into $\mathfrak{f}$, and let H be a graph of the minimal basis $M(\mathfrak{f}, >_1)$. Since H is minimal, properties 1. and 2. are trivially fulfilled. We shall prove 3. indirectly. Suppose, a is a vertex of H which is incident with no edge of H or with one edge k of H or with two edges k_1 and k_2 of H. Obviously $H-a$ or $H-k$ or the graphs obtained from H by contracting k_1 or k_2 will be embeddable into $\mathfrak{f}$. Thus we obtain a contradiction to the minimality of H.

Now assume that H fulfils the properties 1., 2. and 3. Clearly, 1. implies $H\in\Gamma$. Because of theorem (1.1) we find a graph $G\in M(\mathfrak{f}, >_1)$ with $H>_1G$, and $U(G)\subseteqq H$. If H contains an edge k which doesn't belong to $U(G)$, property 2. implies the embedding of $U(G)$ and G. This is a contradiction. Obviously, $H=U(G)$. Since H fulfils property 3., we can conclude that $H=G$. ∎

Before formulating and proving theorem (1.3) we shall give some remarks. At first we say that the orientable surface of the genus $p \in \mathbf{N}_0$ is obtained from sphere S by attaching p handles. Now if S is the sphere again, we can attach $q \in \mathbf{N}$ cross-caps to S by respective removal of the interior of the q closed discs and respective identifiention of the boundary of one of the q Möbius strips with one boundary circle of the q discs. The orientable genus $\gamma(G)$ of a graph G is defined as the minimal genus of an orientable surface into which G can be embedded. In other words $\gamma(G)=p$ means that G can be embedded into the orientable surface of the genus p but not into the surface of the genus $p-1$. In the same way we define the non-orientable genus of a graph. The non-orientable genus $\tilde{\gamma}(G)$ of a graph G is the minimal genus of a non-orientable surface into which G can be embedded. Is $\mathfrak{f}$ an arbitrary orientable surface the following theorem (1.3) classifies the minimal graphs of $M(\mathfrak{f}, >_4)$.

(1.3) *If* $\mathfrak{f}$ *is an orientable surface of genus* $p \in \mathbf{N}_0$ *each graph* G *of* $M(\mathfrak{f}, >_4)$ *has the orientable genus* $\gamma(G)=p+1$ *and the non-orientable genus* $\tilde{\gamma}(G)$ *with* $1 \leqq \tilde{\gamma}(G) \leqq$ $\leqq 2p+2$.

Proof. Let $\mathfrak{f}$ and G be an orientable surface of the genus $p \in \mathbf{N}_0$ respectively a minimal graph of $M(\mathfrak{f}, >_4)$. Property 1. of theorem (1.2) implies $\gamma(G) \geqq p+1$. To finish the proof of the equality $\gamma(G)=p+1$, we need only show that $\gamma(G) \leqq p+1$. Let $k=(a, b)$ be an arbitrary edge of G. The graph $G-k$ is embeddable into $\mathfrak{f}$ because G fulfils the property 2. of theorem (1.2). Now add a "thin" handle for the edge $k=(a, b)$ by attaching both ends of the handle near vertices a and b and avoiding all the rest of G. This handle can be deformed to miss the p handles of $\mathfrak{f}$. Obviously G is embeddable into the orientable surface with $p+1$ handles. Therefore $\gamma(G) \leqq p+1$. For the proof of the inequality we consider the graph $G-k$ with $k=(a, b)$ again. $G-k$ is embeddable into $\mathfrak{f}$. Then we add k by drawing on $\mathfrak{f}$. Without loss of generality we can assume that there is only a finite number of intersections with edges of $G-k$ on $\mathfrak{f}$ and that no vertex e of G with $e \neq a, b$ is incident with k. Let $s_1, s_2, \ldots, s_n$ $(n \in \mathbf{N})$ be the intersections we find if we walk along k from a to b. Now we add a second edge k' from a to b, very close to k on $\mathfrak{f}$ so that we obtain the corresponding intersections $s_1', s_2', \ldots, s_n'$. Then we rip up the surface $\mathfrak{f}$ respectively along k and k' from a point between a and s_1 (s_1') to a point between s_n (s_n') and b so that $\mathfrak{f}$ has two holes. By identifying the borders of those holes like a Möbius-strip we obtain a non-orientable surface $\mathfrak{f}^*$ with p handles and two crosscaps ,and with the non-orientable genus $2p+2$. Obviously we can draw the edge k on $\mathfrak{f}^*$ without intersections with the other edges of G. Therefore G is embeddable into $\mathfrak{f}^*$, so that the non-orientable genus $\tilde{\gamma}(G)$ fulfils the inequality $1 \leqq \tilde{\gamma}(G) \leqq 2p+2$. ∎

Paper [1] displays the simple graphs G_p $(p \geqq 1)$ with $\gamma(G_p)=p$ and each can be embedded into the projective plane. Because of theorem (1.2) and the embedding-heriditariness of the subdivision $>_1$ we are able to conclude that the projective plane contains graphs of the minimal basis $M(\mathfrak{f}, >_1)$ for every orientable surface $\mathfrak{f}$ (that is to say for every p). The problem whether we can always find minimal graphs of $M(\mathfrak{f}, >_1)$ with the non-orientable genus $q=2p+2$ or whether q is always less than or equal to $p+1$ remains questionable. But it's obviously clear that the following specialization of theorem (1.3) is right.

(1.4) *Let* $\mathfrak{f}$ *be an orientable surface of genus* p. *If a graph* H *of* $M(\mathfrak{f}, >_1)$ *can be drawn on* $\mathfrak{f}$ *with only one intersection of two edges, the nonorientable genus* $\bar{\gamma}(H)$ *is less than* $2p+1$. ∎

Now we should like to specify all minimal graphs of M(torus, $>_1$) having the non-orientable genus $q=1$ and therefore being embeddable into the projective plane. Solving this problem the so-called "Rhön"-wheels" of the projective plane are displayed. Two Rhön-wheels of the sphere with eight vertices are drawn in Figure 1. The figures 2 and 3 represent

Fig. 1

two Rhön-wheels X_8 and X_7 of the projective plane with eight,

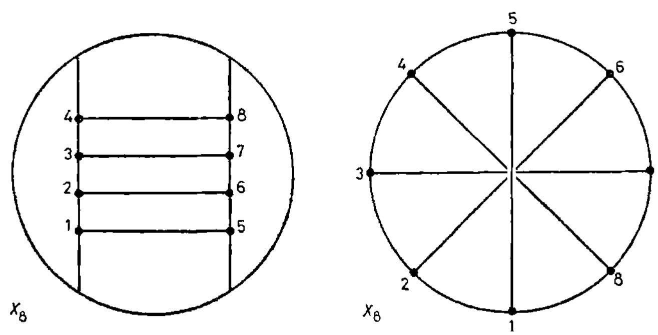


Fig. 2

respectively seven vertices[3].

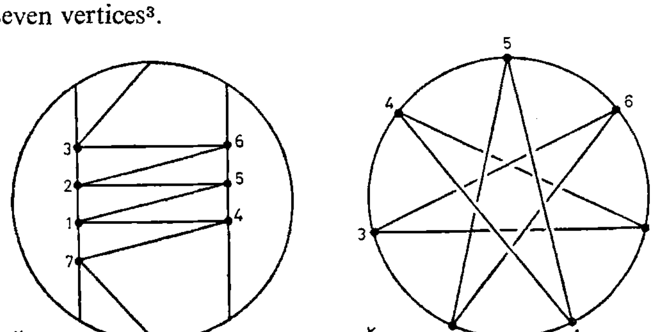


Fig. 3

[3] In another context of [1] it has been proved that H_1 can be seen as an example for a graph in the projective plane which is not embeddable into the torus.

An embedding of X_8 into the projective plane is shown on the left of Figure 2. On the right of Figure 2 we find another usual embedding of X_8 with a greater number of intersections of edges. Figure 3 represents a Rhön-wheel X_7 with seven vertices on the right. Rhön-wheels of the projective plane always have an odd number of vertices. On the left of Figure 3 we find an embedding of X_7 into the projective plane. If we join together the circuit $(1, 2, \ldots, 8, 1)$ of X_8 respectively the circuit $(1, 2, \ldots \ldots, 7, 1)$ of X_7 and the Rhön-wheel we obtain the two graphs H_1 and H_2 being represented in Figure 4. H_1 and H_2 are embeddable into the projective plane as we can see in Figures 2 and 3. Since both of them are not planar H_1 and H_2 have non-orientable genus 1. Embedding X_8 into the projective plane (Figure 2) we see that there are four quadrilaterals and an octagon. Now we prove the following:

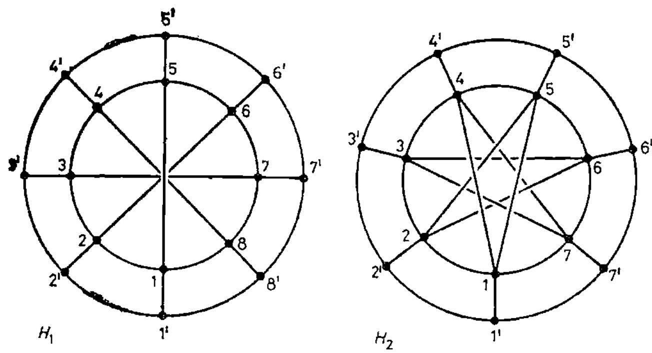


Fig. 4

Theorem 1. *The graphs H_1 and H_2 being introduced in Figure 4 belong to the minimalbasis $M(\Gamma_t, >_1)$ where Γ_t is the set of all graphs not embeddable into the torus.*

Proof. Refering to theorem (1.2) we must at first show that H_1 and H_2 are not embeddable into the torus. This will be proved indirectly. We assume that H_1 and H_2 are embeddable into the torus. Then X_8 and X_7 being subgraphs of H_1 and H_2 are embeddable into the torus. We obtain X_8 by subdividing of the two edges $(3, 5)$ and $(1, 7)$ of the $K_{3,3}$ with the two vertex-triples $1, 3, 6$ and $2, 5, 7$ and adding the edge $(4, 8)$. The points 4 and 8 are the new vertices on the edges $(3, 5)$ and $(1, 7)$. Corresponding to this method the vertex 7 is added on the edge $(1, 6)$ of the $K_{3,3}$ with the two vertex-triples $1, 3, 5$ and $2, 4, 6$ and then the edges $(1, 5)$, $(2, 6)$, $(3, 7)$ and $(7, 4)$ are added, too. In order to obtain the required contradiction we shall apply to the fact that the $K_{3,3}$ is only embeddable into the torus in two topologically distinct ways A and B (Figure 5). Therefore having subdivided the edges (3,5)

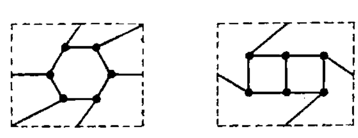


Fig. 5

and (1, 7) or respectively the edge (1, 6) is must be possible to embed the missing edges (4, 8) or respectively (1, 5), (2, 6), (3, 7), (7, 4) and the circuit (1′, 2′, ..., 8′, 1′) and the eight edges (1, 1′), (2, 2′), ..., (8, 8′) or respectively the circuit (1′, 2′, ..., 7′, 1′) and the seven edges (1, 1′), (2, 2′), ..., (7, 7′) into the torus. As there are intersections of edges in any case we obtain the required contradiction. Now we can assume that H_1 and H_2 are not embeddable into the torus. In order to prove that H_1-k and H_2-k' are embeddable into the torus for each edge k of H_1 and k' of H_2 we only have to consider the following four edges $k=(1, 1'), (1, 5), (1, 2)$ and $(1', 2')$ or respectively $k'=(1, 1'), (1, 4), (1, 2)$ and $(1', 2')$. This is based on the symmetry of H_1 and H_2. Figure 6 shows the embeddings of $H_2-(1, 1')$, $H_2-(1, 4)$, $H_2-(1, 2)$ and $H_2-(1', 2')$ into the torus. Corresponding to that we obtain the embeddings of $H_1-(1, 1')$, $H_1-(1, 5)$, $H_1-(1, 12)$ and $H_1-(1', 2')$ into the torus. Since the degree of each vertex of H_1 and H_2 is greater than or equal to

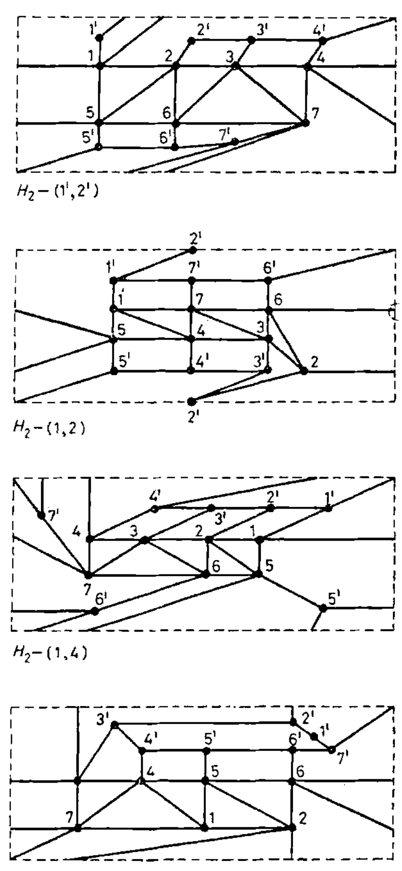


Fig. 6

three the proposition $H_1, H_2 \in M(\text{torus}, >_1)$ now follows immediately from theorem (1.2). ▮

A characterization of the minimal graphs of $M(\text{torus}, >_1)$ with nonorientable genus $q=1$ is given in

Theorem 2. *Let H be a graph of $M(\text{torus}, >_1)$ with non-orientable genus $q=1$. Then H contains an X_8 or an X_7 for every embedding into the projective plane with the following properties:*

1. At most the edges of the octogon of X_8 respectively of the heptagon of X_7 are subdivided.

2. There are at most the edges of H in the interior of the four countries of X_8 which are bordered by a quadrilateral. Each of these edges is joining a vertex of the "left" subdivided edge with a vertex of the "right" subdivided edge of a quadrilateral (Figure 2). Corresponding to that there are no vertices and no edges of H in the interior of the seven countries of X_7 which are bordered by a triangle.

3. All the other vertices and edges of H are in the octogon of X_8 respectively in the heptagon of X_7.

Proof. Let H be a graph of $M(\text{torus}, >_1)$ with non-orientable genus $q=1$ and $X_8 \nsubseteq H$. At first we shall prove that H contains an $U(K_{3,3})$. Suppose that H doesn't contain an $U(K_{3,3})$. Then H contains an $U(K_5)$. Since H is minimal $U(K_5)=K_5$. Then H must contain another $K_5' \neq K_5$, for the K_5 is embeddable into the torus. The intersection graph of K_5 and K_5' doesn't contain more than two vertices. Otherwise we would obtain an $U(K_{3,3})$. But as we know from [2, 3] the union graph of K_5 and K_5' is not embeddable into the projective plane. This is a contradiction to H having the non-orientable genus $q=1$. So we can assume that H contains an $U(K_{3,3})$. Now we consider a determined embedding of H into the projective plane. The subgraph $U(K_{3,3})$ of H divided the projective plane in three countries, which are bordered by a quadrilateral, and one country, which is bordered by a hexagon. Now we call a connected subgraph of H, which is embeddable into one of the three quadrilateral of $U(K_{3,3})$, inner relative component in relation to $U(K_{3,3})$. A connected subgraph of H, which is embeddable into the hexagon of $U(K_{3,3})$ is called outer relative component in relation to $U(K_{3,3})$. At last relative components consisting of only one edge are called diagonal edges. It may happen that we find another "smaller" quadrilateral in a quadrilateral of the $U(K_{3,3})$ which delivers another subdivision of $K_{3,3}$ together with the other two quadrilaterals of the $U(K_{3,3})$ of H. If we continue this method of determining new subdivisions of $K_{3,3}$ in H we obtain an $\hat{U}(K_{3,3}) \subseteq H$ at last for which each inner relative component in relation to $\hat{U}(K_{3,3})$ is a diagonal edge, which respectively joins one vertex of the "left" subdivided edge with one vertex of the "right" subdivided edge of the quadrilateral of the $\hat{U}(K_{3,3})$. Besides each of the three main trails of the $\hat{U}(K_{3,3})$ separating two quadrilateral of $\hat{U}(K_{3,3})$ must be an edge, for H is minimal. Therefore these three edges and all the inner diagonal edges are briefly called the inner edges of H relative to $\hat{U}(K_{3,3})$. If the graph H contained four vertexdisjoint edges, H would be a graph performing all the three properties of theorem 2. Therefore we can assume that each inner edge of H is incident to a main vertex of $\hat{U}(K_{3,3})$. All the inner diagonal edges of H can consequently be divided into 12 different bundles of edges $B_1, B_2, \ldots, \ldots, B_{12}$, as shown in Figure 7. Each bundle contains all the diagonal edges being

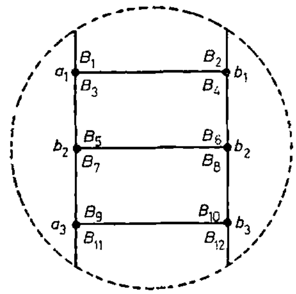


Fig. 7

incident to the main vertex of $\hat{U}(K_{3,3})$ referred to and lying in the quadrilateral $\hat{U}(K_{3,3})$ referred to. If i is odd with $1 \leqq i \leqq 11$ one of the bundles B_i and B_{i+1} is an empty set for there are no intersections of edges of H (Figure 7). Embedding the $\hat{U}(K_{3,3})$ into the torus like A in Figure 5 and denoting the main vertices of the hexagon in the middle of embedding A in the same sequence as in Figure 7 is decisive for the following proof. All the outer relative components of H can be embedded into the hexagon of the middle of A. Since H is not embeddable into the torus we cannot embed all the bundles $B_1, B_2, \ldots, B_{12}$ into the two other hexagons of A. We have tried to draw all these bundles of edges into the two other hexagons (Figure 8). It had to be taken into consideration that there mustn't be an intersections of two "neighbouring" bundles of Figure 7 (for example B_3 and B_5 or B_7 and B_9). Since H isn't embeddable into the torus there are intersections of at least two bundles. Without loss of generality we can assume intersections of B_2 and B_3 with

(1) $$B_2 \neq \emptyset \quad \text{and} \quad B_3 \neq \emptyset.$$

Since H doesn't contain four vertex disjoint inner edges it is clear that $B_2 \supseteqq \{(b_1, b_3)\}$ or $B_3 \supseteqq \{(a_1, a_2)\}$. Now let be $B_2 = \{(b_1, b_3)\}$. Then the equality

(2) $$B_1 = B_4 = B_{11} = \emptyset$$

follows immediately. Now we shall have to discuss the two possibilities

(α) $$B_3 = \{(a_1, a_2)\} \quad \text{and} \quad (\beta) \quad B_3 \supsetneqq \{(a_1, a_2)\}.$$

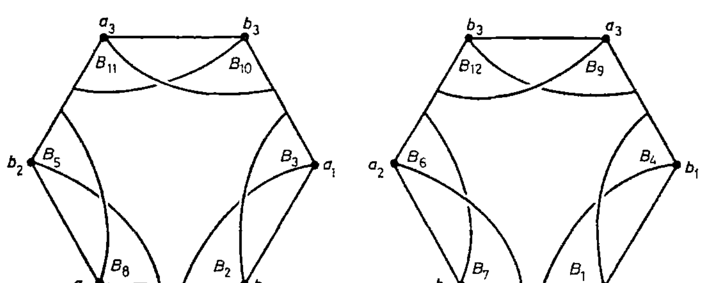


Fig. 8

At first $B_5=\emptyset$ follows from (α). If $B_7=\{(b_2, b_3)\}$ or $B_9=\{(a_2, a_3)\}$, H would be embeddable into the torus (Figures 7 and 8). Corresponding to this H would be embeddable into the torus if $B_7=B_9=\emptyset$. If $B_7=\emptyset$ and $B_9\neq\emptyset$ and so $B_{10}=\emptyset$, B_9 contains an edge $\neq(a_2, a_3)$ and $B_{12}=\{(b_1, b_3)\}$. Otherwise H would contain four vertex disjoint edges. But then H would be embeddable into the torus. Corresponding to this we can derive a contradiction in case of $B_9=\emptyset$ and $B_7\neq\emptyset$. So we have $B_7\neq\emptyset$, $B_9\neq\emptyset$, $B_7\neq\{(b_2, b_3)\}$ and $B_9=\{(a_2, a_3)\}$. Then B_7 and B_9 only contain one edge each and both edges are incident to the same vertex of the main path $a_2 \dots b_3$ of $U(K_{3,3})$ for H would have four vertex-disjoint inner edges otherwise. Obviously H now fulfils the proposition of X_7. In case of (β) B_3 contains an edge $\neq(a_1, a_2)$. Furthermore $(b_1, b_2)\in B_{12}$. Since H doesn't contain four vertex-disjoint edges either $B_9=\emptyset$ or $B_9=\{(a_2, a_3)\}$. Corresponding to the last possibility of (α) we can prove the proposition of X_7 in case of $B_9=\{(a_2, a_3)\}$. Therefore we can assume

(3) $$B_9 = \emptyset.$$

Since H isn't embeddable into the torus there must be an intersection of either B_6 and B_7 or B_5 and B_8 (Figures 7 and 8). Then one of these bundles must be the diagonal. If $B_6=\{(a_1, a_2)\}$ or $B_7=\{(b_2, b_3)\}$, H would be embeddable into the torus. If $B_5=\{(b_1, b_2)\}$ then $B_3=\emptyset$, in contradiction to (1). In case of $B_8=\{(a_2, a_3)\}$ we have the equality

(4) $$B_7 = B_{10} = \emptyset \quad \text{with} \quad B_5 \neq \emptyset,$$

for H isn't embeddable into the torus. Corresponding to $B_9=\{(a_2, a_3)\}$ we obtain the proposition of X_7, and all cases have been discussed. ▌

Determining the minimal basis of all graphs not being embeddable into the projective plane we see from [2] and [3] that it is very useful to intensity the propositions of the minimal basis. Therefore the partial order relations $>_2$, $>_3$ and $>_4$ were introduced. Referring to the partial order relations $>_1$, $>_2$, $>_3$, $>_4$ every minimal basis of a set of graphs $\Gamma\neq\emptyset$ fulfils the proposition $M(\Gamma, >_1)\supseteqq M(\Gamma, >_2)\supseteqq$ $\supseteqq M(\Gamma, >_3)\supseteqq M(\Gamma, >_4)$. Now theorem (1.2) can be a supplemented by the following properties 4., 5. and 6.:

4. Each $R_2(H)$ is embeddable into $\mathfrak{f}$.
5. Each $R_3(H)$ is embeddable into $\mathfrak{f}$.
6. Each $R_4(H)$ is embeddable into $\mathfrak{f}$.

Corresponding to theorem (1.2) the theorem (1.2*) is worth pointing out

(1.2*) *A graph H belongs to $M(\mathfrak{f}, >_i)$ with $i=1, 2, 3, 4$ if, and only if, H fulfils the properties* 1., 2., ..., $(i+2)$. ▌

Since (1.2) has been proved in full length we renounce the proof of (1.2*). Applying to (1.2*) and the embeddings A and B of the $K_{3,3}$ into the torus we shall now prove the following theorem

(1.5) *The fourteen graphs $G_1, G_2, \dots, G_{14}$ of Figure 9 belong to* $M(\text{torus}, >_4)$.

Proving theorem (1.5) we decisively use the fact that each graph H of M(torus, $>_4$) with $H\neq G_1, G_2, G_3, G_4, G_5$ contains a $U(K_{3,3})$. Since we have to discuss a lot of different possibilities we renounce the further proof.

With the help of the 12 minimal gaphs of M(projective plane, $>_4$) in the first Figure of [2] we can decide that each of the 14 graphs of Figure 9 has the non-orientable genus $q \geqq 2$.

In order to characterize the non-orientable genus of a graph G we at first introduce the concept of the "diagonal cross point". It's clear that every graph G can be drawn in the plane if crossing of edges are admitted. If it's possible to draw all the edges going through a cross point as straight lines in a sufficiently small neighbourhood of this cross point, we call it a diagonal cross point of the graph G. For example there is the graph G_8 of Figure 9. It's obviously clear that every crossing of exactly two edges always is a diagonal cross point.

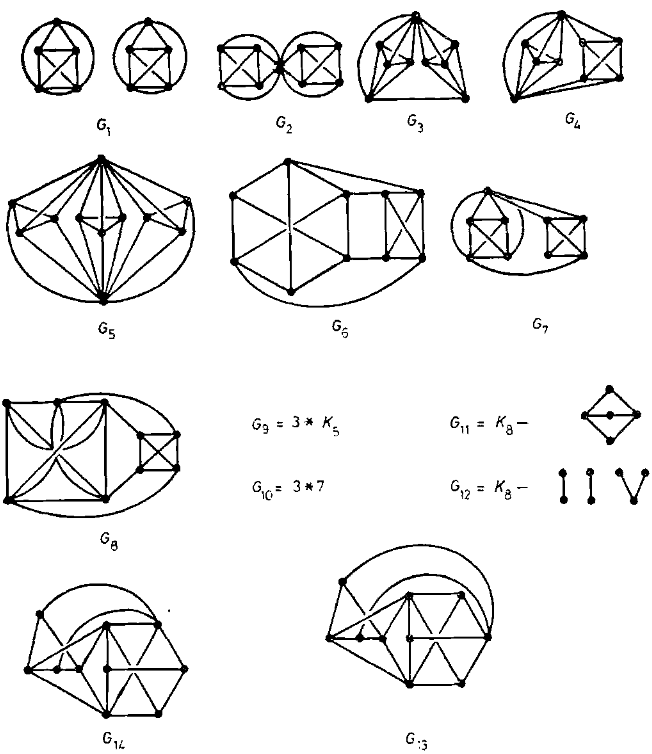


Fig. 9

Theorem 3. *If G is a graph of the non-orientable genus $q \geqq 1$, G can always be drawn as a graph in the plane with diagonal cross points in such a manner that there are q diagonal cross points, and there doesn't exist a drawing of G in the plane with less than q diagonal cross points.*

Proof. Let G be a graph of the non-orientable genus $q \geqq 1$. Then G can be drawn in the sphere with q crossing caps. Drawing the edges on each crossing cap diagonally we always obtain a diagonal cross point of G. Therefore G can be drawn in the plane as a graph with q diagonal cross points. On the other hand if G can be drawn in the plane with only d diagonal cross points it follows that the non-orientable genus q is less than or equal to d with $q \leqq d$. Therefore $d=q$. ▮

Since all 14 graphs of Figure 9 have a non-orientable genus $q \geqq 2$ we get

Theorem 4. *If G is one of the graphs G_1, G_2, G_3, G_4, G_6, G_7, G_8, G_{11}, G_{13}, G_{14} of Figure 9, the non-orientable genus $\gamma(G)=2$.*

Proof. Applying to theorem 3 the proposition of theorem 4 follows immediately with the exception of G_{11} (Figure 9). G_{11} is obtained from a K_8 by deleting all the six edges of the complete bipartite graph $2*3$ is composed of a K_6 and a K_2 at which both vertices of the K_2 are adjacent to the three vertices of a triangle of the K_6. Since K_6 is embeddable into the projective plane and every embedding of K_6 triangulates the projective plane it follows from theorem 3 that G_{11} has a non-orientable genus $q=2$. ▮

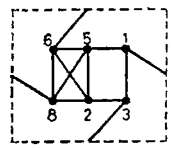
Theorem 5. *The graphs G_5, G_9, G_{10} and G_{12} of M(torus, $>_4$) (Figure 9) have a non-orientable genus $q=3$.*

Proof. Looking at G_5 we find out that for every embedding of G_5 into the plane each of the three K_5-k forming G_5 has at least one cross point. Since each graph with n vertices and k edges embedded into Klein's bottle fulfils the estimation $k \leqq 3n$ and G_9 has $n=8$ vertices and $k=25$ edges, G_9 has the non-orientable genus $q \geqq 3$. On the other hand applying to an embedding of K_7 into the torus G_9 can easily be drawn in the torus with only one crossing of two edges. Since the torus with one crossing cap added is equivalent to a non-orientable surface of genus 3, it follows that G_9 has non-orientable genus $q \geqq 3$, and therefore $q=3$. Referring to embedding A of $K_{3,3}$ into the torus (Figure 5) we find out that $G_{10}=K_{3,7}=3*7$ has also non-orientable genus $q \leqq 3$. But in case of G_{10} we can't prove the proposition by using the estimation $k \leqq 3n$. In order to show $q \geqq 3$ several possibilities must be discussed. Since some of them can be found in [2] and [3], we renounce the proof of this property now. In case of K_{12} we have to proceed similarly. In order to prove $q \leqq 3$ we look at Figure 10 showing that G_{12} obtained from the K_8 with the vertices $1, 2, \ldots, 8$ by deleting the edges (1, 2), (3, 4), (4, 5), (6,7) can be drawn in the torus with only one crossing of the two edges (2, 6) and (5, 8) of G_{12}.

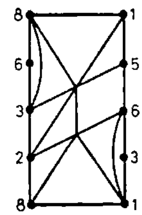
Now we shall discuss the case of the non-orientable genus $q=1$ for H_1 and H_2 being introduced in Figure 4 have non-orientable genus $q=1$. As we know from theorem 1 H_1 and H_2 belong to M(torus, $>_1$). It's quite astonishing that we obtain five graphs G_{15}, G_{16}, G_{17}, G_{18} and G_{19} of M(torus, $>_4$) with non-orientable genus $q=1$ only by applying theorem (1.2*) (Figure 11).

Theorem 6. *The graphs $G_{15}, G_{16}, G_{17}, G_{18}$ and G_{19} of non-orientable genus $q=1$ (Figure 11) belong to M(torus, $>_4$).*

Proof. At first we shall prove that H_1 and H_2 even belong to M(torus, $>_2$). Because of theorem 1 and theorem (1.2*) it remains to show that all graphs obtained from H_1 and H_2 by contracting an edge k of H_1 or H_2, are always embeddable into the torus. From the symmetry of H_1 and H_2 it's clear that we only have to discuss three possibilities at a time. The embeddings into the torus of these six graphs follow immediately from embeddings A and B of the $K_{3,3}$ (Figure 5). Applying (1.2*) and minimalizing H_1 and H_2 with R_3 and R_4 (graphically at the vertices of the exterior



B



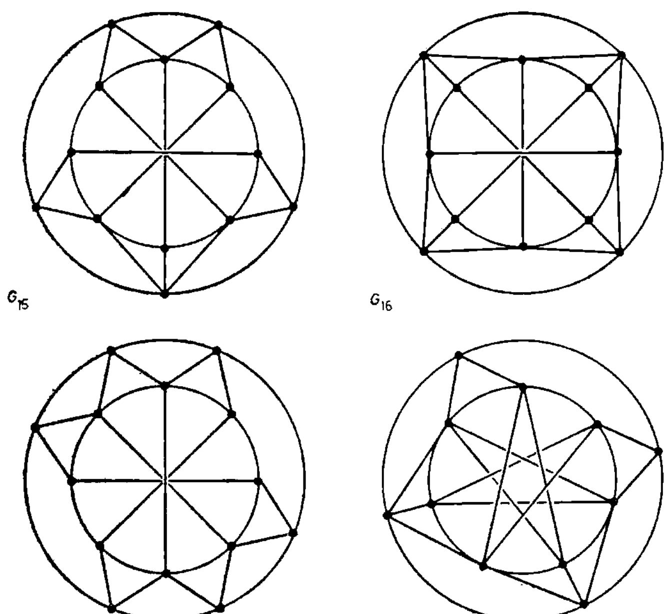


Fig. 10

G_{19}

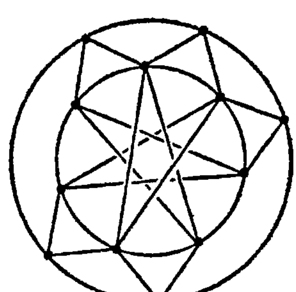


Fig. 11

circuit of H_1 and H_2 in Figure 4) we obtain the five graphs $G_{15}-G_{19}$. Specifying this we can say: $G_{15}=R_3^3(H_1)$, $G_{16}=R_3^4(H_1)$, $G_{17}=R_4^2(H_1)$, $G_{18}=R_3^3(H_2)$ and $G_{19}=R_3(R_4(H_2))$. Since each graph $R_{i+1}(G)$ $(i=1,2,3)$ of a graph $G\in M(\mathfrak{f}, \succ_i)$ (for each orientable or non-orientable surface $\mathfrak{f}$) is always embeddable into $\mathfrak{f}$ or belongs to $M(\mathfrak{f}, \succ_i)$ the proposition of theorem 6 is proved[4].

So we can summarize that $G_1, G_2, \ldots, G_{19}$ represent a number of interesting graphs of $M(\text{torus}, \succ_4)$ with the non-orientable genus $q=1, 2$ or 3. It is not known whether we can find graphs of $M(\text{torus}, \succ_4)$ with non-orientable genus $q=4$. From theorem 2 we can see that the graphs of $M(\text{torus}, \succ_1)$ and because of $M(\text{torus}, \succ_4)\subseteq M(\text{torus}, \succ_1)$ the graphs of $M(\text{torus}, \succ_4)$ can be classified corresponding to X_7 or X_8. Let G be a graph of $M(\text{torus}, \succ_4)$ containing a X_7. Since $\gamma(G)=1$ we can regard a fixed embedding of G into the projective plane. Applying theorem 2 we know that all relative components of G in relation to X_7 (or "Querstücke" of G in relation to X_7) excepted the four edges of X_7 not belonging to the $U(K_{3,3})\subseteq X_7$ can be found in the heptagon S_7 of X_7 (Figure 3). The seven paths 1...2, 2...3, 6...7 and 7...1 of S_7 are shortly called the sides of S_7. Furthermore it's said that the side 3...4 is opposite to the sides 6...7 and 7...1, and similarly for the other sides of S_7.

(1.6) *If p and q are vertices on two sides being opposite*[5] *to each other every path from p to q within S_7 (in the projective plane) crosses at least one relative component of G in relation to X_7.*

Proof. Because of the symmetry of X_7 we can assume that p is lying on side 1...2 and q is lying on side 4...5. Suppose that path W doesn't cross any relative component Q of G in relation to X_7. Since W divides the interior of S_7 into two components S' and S'' every relative component Q of G relation to X_7 is lying in S' or S''. Figure 12 shows an embedding of X_7 without edge (1, 5) into the torus on the left. On the right you see an embedding of X_7 into the torus where edge (1, 5) has been drawn into the decagon. S' and S'' have been hatched in Figure 12. As obviously every relative component Q of G in relation to X_7 can be embedded into the decagon, we have found a contradiction to G not being embeddable into the torus. ∎

It follows from (1.6) that all basis points of a relative component Q of G in relation to X_7 are not lying on one side of S_7 and on both sides being opposite.

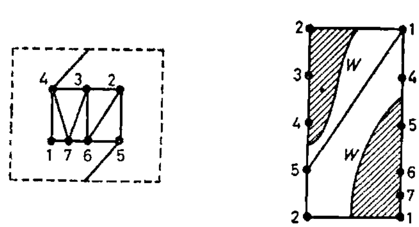


Fig. 12

[4] See [4].

[5] The reader should note that the endpoints of the two sides are included and on the hand p and q are not to be thought of lying on the same side of S_7.

Therefore theorem (1.6) shows how we can determine all possible relative components of G in relation to X_7 and besides all minimal graphs with non-orientable genus $q=1$. This method can be called as "the satisfying method". It can be discribed graphically in the following way: Embed the X_7 into the torus and add possible relative components of G in relation to X_7 until you must leave the torus. Such a satisfying method has already been used in order to specify the 12 minimal graphs of M(projective plane, $>_4$) in [2] and [3]. This satisfying method now can similarly be used for the other minimal graphs of M(torus, $>_4$) with the non-orientable genus $q>1$. Before describing that way some remarks are necessary. Let G be a graph of M(torus, $>_4$) with the non-orientable genus $q>1$. Since the minimal basis M(projective plane, $>_4$) is well known—from [2] and [3] we known all the minimal graphs[6] $G'_1, G'_2, \ldots, G'_{12}$—it follows that $G>_4 G_\nu$ for at least one $\nu=1, 2, \ldots, 12$. Because of $G'_1=G_1$ and $G'_2=G_2$ $\nu=1$ and $\nu=2$ can be omitted.

Based on the well known properties of the four minimal bases M(projective plane, $>_i$) $(i=1, 2, 3, 4)$ we can find a graph $G^{(i)}\in M$(projective plane, $>_i$) for each $i=1, 2, 3, 4$ with $G>_i G^{(i)}$. Obviously $G^{(4)}=G'_\nu$ for at least one $\nu\in\{3, 4, \ldots, 12\}$. If $G^{(i)}\neq G^{(i-1)}$ $(i=2, 3, 4)$ we can assume a sequence of graphs $G_0^{(i)}, G_1^{(i)}, \ldots, G_{\lambda_i}^{(i)}$ with $G_0^{(i)}=G^{(i-1)}$, $G_{\lambda_i}^{(i)}=G^{(i)}$ and $G_\mu^{(i)}=R_i(G_{\mu-1}^{(i)})$ $(\mu=1, 2, \ldots, \lambda_i)$.

(1.7) *Let G be a graph of M*(torus, $>_4$) *with the non-orientable genus $q>1$. Then there are four graphs $G^{(i)}\in M$*(projective plane, $>_i$) $(i=1, 2, 3, 4)$ *with the following properties*:
1. $G>_1 G^{(1)}$.
2. $G^{(i)}=R_i^{\lambda_i}(G^{(i-1)})$ *for*[7] *each* $i=2, 3, 4$, *and*
3. M(projective plane, $>_4$)$=\{G'_1, G'_2, \ldots, G'_{12}\}$.

Applying to theorem (1.7) we now can describe the "satisfying method" for graphs with non-orientable genus $q>1$. Coming from each G'_ν of M(projective plane, $>_4$) $(\nu=3, 4, \ldots, 12)$ all the graphs $R_4^{-1}(G')$, $R_4^{-2}(G'_\nu)$, $R_4^{-3}(G'_\nu), \ldots$ are to be concerned until we first obtain a graph $R_4^{-\lambda}(G'_\nu)$ $(\lambda=1, 2, 3, \ldots)$ not being embeddable into the torus for $G'_3, G'_4, \ldots, G'_{12}$ are embeddable into the torus. Simplifying the description we shall call all those graphs R_4^{-1}-critical which are not embeddable into the torus. The set of the non-R_4^{-1}-critical graphs will be called R_4^{-1}-closure of $\{G'_3, G'_4, \ldots, G'_{12}\}$ in the torus and will be denoted by $H_4^{-1}(\{G'_3, G'_4, \ldots, G'_{12}\}, p=1)$ or shortly by H_4^{-1}. Similarly we regard all the R_3^{-1}-critical graphs of H_4^{-1} and the R_3^{-1}-closure of H_4^{-1} in the torus. It's shortly denoted by $H_{3,4}^{-1}$. At last we have to consider all R_2^{-1}-critical graphs of $H_{3,4}^{-1}$. The R_2^{-1}-closure of $H_{3,4}^{-1}$ in the torus is denoted by $H_{2,3,4}^{-1}$. So we obtain two finite sets of graphs denoted by $\mathfrak{K}(\{G'_3, G'_4, \ldots, G'_{12}\}, p>1)$ and $H^{-1}(\{G'_2, G'_4, \ldots, G'_{12}\}, p=1)=H_4^{-1}\cup \cup H_{3,4}^{-1}\cup H_{2,3,4}^{-1}$. The first set is the set of all critical graphs of orientable genus $p>1$.

(1.8) *Let G be a graph of M*(torus, $>_4$) *with non-orientable genus $q>1$. Then either $G\in\mathfrak{K}(\{G'_3, G'_4, \ldots, G'_{12}\}, p>1)$ or G contains a subdivision of at least one graph of $H^{-1}(\{G'_3, G'_4, \ldots, G'_{12}\}, p=1)$.*

[6] The 12 graphs of M(projective plane, $>_4$) are denoted with apostroshies in order to differ from $G_1, G_2, \ldots, G_{12}$ of M(torus, $>_4$).

[7] λ_i means a natural number, 0 included. In case of $\lambda_i=0$ the above menioned equation is $G^{(i)}=G^{(i-1)}$.

Proof. From (1.7), (2) we know that there is a $G^{(3)}\in M$(projective plane, $>_3$) with $G^{(4)}=R_4^{\lambda_4}(G^{(3)})$ and $G^{(4)}=G'_v$ for $v\in\{3, 4, \dots, 12\}$. It follows that $G^{(3)}=R_4^{\lambda_4}(G'_4)$. In case of $\lambda_4=0$, $G^{(3)}=G^{(4)}$. So we only need to regard $G^{(3)}$. Applying (1.7) we can conclude similarly in case of $i=3$ and $i=2$, and we obtain $G^{(1)}=R_2^{-\lambda_2}(R_3^{-\lambda_3}(R_3^{-\lambda_3}(G')))$ with $G>_1G^{(1)}$. ▮

The last equation of the proof of theorem (1.8) graphically shows a simple method of construction the graph $G^{(1)}$. Coming from G'_v $(v=3, 4, \dots, 12)$ we obtain $G^{(1)}$ by using R_4^{-1}, R_3^{-1} and R_2^{-1} retrogradely. Since all the graphs G of M(torus, $>_4$) excepted G_1 and G_2 (Figure 11) are at least 2-connected, and G_1, G_2 may be excluded in the above mentioned consideration we obtain another construction $G>_1G^{(1)}$ from $G^{(1)}$. We add relative components to $G^{(1)}$ in such a manner that each relative component is attached to $G^{(1)}$ in at least two basis points of $G^{(1)}$. Doing this we must observe the facts that all the relative components excepted their basis points must be disjoint, and that all the basis points are either vertices of $G^{(1)}$ or new vertices added or edges of $G^{(1)}$. We state this result formally in the following

Theorem 7. *If* $G\neq G_1, G_2$ *is a graph of* M(torus, $>_4$) *with non-orientable genus* $q>1$, G *can be built from a graph* $G^{(1)}=R_2^{-\lambda_2}(R_3^{-\lambda_3}(R_4^{-\lambda_4}(G_v)))$ $(v\in\{3, 4, \dots, 12\})$ *and relative components in relation to* $G^{(1)}$ *with at least two basis points each in such a manner that all the basis points are vertices of* $G^{(1)}$ *or new vertices on edges of* $G^{(1)}$ *and that all the relative components excepted their basis points are disjoint to each other and to* $G^{(1)}$. ▮

We now conclude with an example showing how we can obtain the graph G_{20} (Figure 13) as a new graph of M(torus, $>_4$) using theorem 7.

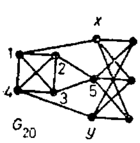


Fig. 13

(1.9) $G_{20}\in M$(torus, $>_4$).

Proof. Since G_{20} contains an $U(K_{3,3})$ which divides the torus into 2-cells for every embedding into the torus G_{20} is not embeddable into the torus. Furthermore we see immediately that we can't use R_4 in case of G_{20} and that we can use R_3 in case of G_{20} in exactly one way. Figure 14 shows $R_3(G_{20})$ and an embedding of $R_3(G_{20})$ into the torus on the right. We realize that we can't use R_2 in case of eight of the 19 edges of

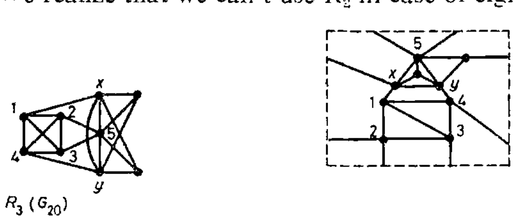


Fig. 14

the G_{20}. Because of the symmetry of G_{20} there are only three edges to be considered. Applying to the embeddings A and B of the $K_{3,3}$ into the torus (Figure 5) we can prove analogically to the graphs $R_3(G_{20})$ that the three different $R_3(G_{20})$ are embeddable into the torus. $G_{20} \in M(\text{torus}, >_4)$ follows from (1.2*). ∎

Since G_{20} isn't embeddable into the projective plane and therefore has non-orientable genus $q>2$ G_{20} is suitable for demonstrating the satisfying method in order to construct minimal graphs of $M(\text{torus}, >_4)$. For example the graph $G_6 \in M(\text{torus}, >_4)$ (Figure 9) can be built from $R_3^{-1}(G'_{12})$ by adding a relative component $Q=1*3$ with three basis points. Therefore $G_6=R_3^{-1}(G'_{12}) \cup (1*3)$.

Applying to the definition of the R_4^{-1}-closure we obtain the equation

$$H^{-1}(\{G'_3, G'_4, \ldots, G'_{12}\}, \quad p=1) = \bigcup_{\nu=3}^{12} H^{-1}(\{G'_\nu\}, p=1).$$

Applying to the definition of the critical graphs we obtain the equation

$$\mathfrak{K}(\{G'_3, G'_4, \ldots, G'_{12}\}, p>1) = \bigcup_{\nu=3}^{12} \mathfrak{K}(\{G'_\nu\}, p>1).$$

For example looking at the graphs $G'_5=2*3*3$ and $G'_6=1*3*3*1$ of $M(\text{projective plane}, >_4)$ we obtain $H^{-1}(\{G'_\nu\}, p=1)=H_2^{-1}(\{G'_\nu\}, p=1)$ with $\nu=5, 6$ for they both don't contain triangles or double-triangles. With other words: Constructing the closures of G'_5 and G'_6 we only use the relation R_2. If we remove the edges (1, 4) and (2, 3) of K_4 being contained in G_{20} (Figure 13) and if we then contract the edges $(1, x)$ and $(4, y)$ (Figure 13), we obviously obtain the graph $G'_5= =2*3*3$. This means that the graph G_{20} can be built from the graph $R_2^{-2}(G'_5)$ by adding two relative components. In this case the two relative components are both edges whose basis points are vertices of the $R_2^{-1}(G'_5)$. So in this case we needn't subdivide edges of the graph $R_2^{-2}(G'_5)$.

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